by omitting 1, 2, 3, 4, etc. of the first terms and multiplying the sum of the following terms by the first factor of the denominator of the first term that is retained and by $P$.

49. Now, this sequence of differences is more convergent than a decreasing geometric progression (§§34, 35). Hence the residues $R'$, $R''$, $R'''$, etc. decrease in such a way that they become smaller than any assignable quantity. And as every one of these residues, having $D$ as common divisor, is a multiple of $D$, it follows that this common divisor $D$ is smaller than any assignable quantity, which makes $D = 0$. Consequently $M:P$ is a quantity incommensurable with unity, hence irrational.

50. Hence *every time that a circular arc* $= \varphi/\omega$ *is commensurable with the radius* $= 1$, *hence rational, the tangent of this arc will be a quantity incommensurable with the radius, hence irrational. And conversely, every rational tangent is the tangent of an irrational arc.*

51. Now, since the tangent of 45° is rational, and equal to the radius, the arc of 45°, and hence also the arc of 90°, 180°, 360°, is incommensurable with the radius. Hence *the circumference of the circle does not stand to the diameter as an integer to an integer*. Thus we have here this theorem in the form of a corollary to another theorem that is infinitely more universal.

52. Indeed, it is precisely this absolute universality that may well surprise us.

---

Lambert then goes on to draw consequences from his theorem concerning arcs with rational values of the tangent. Then he draws an analogy between hyperbolic and trigonometric functions and proves from the continued fraction for $e^u + 1$ that $e$ and all its powers with integral exponents are irrational, and that all rational numbers have irrational natural logarithms. He ends with the sweeping conjecture that "no circular or logarithmic transcendental quantity into which no other transcendental quantity enters can be expressed by any irrational radical quantity," where by "radical quantity" he means one that is expressible by such numbers as $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt{2 + \sqrt{3}}$, and so forth. Lambert does not prove this; if he had, he would have solved the problem of the quadrature of the circle. The proof of Lambert's conjecture had to wait for the work of C. Hermite (1873), and F. Lindemann (1882). See, for instance, H. Weber and J. Wellstein, *Encyklopädie der Elementar-Mathematik* (3rd ed.; Teubner, Leipzig, 1909), I, 478–492; G. Hessenberg, *Transzendenz von e und π* (Teubner, Leipzig, Berlin, 1912); U. G. Mitchell and M. Strain, "The number $e$," *Osiris 1* (1936), 476–496.

## 18 FAGNANO AND EULER. ADDITION THEOREM OF ELLIPTIC INTEGRALS

Count Giulio Carlo de'Toschi di Fagnano (1682–1766), Spanish consul in his home town of Sinigaglia (Italy) and an amateur mathematician, published in the *Giornali de'letterati d'Italia* for the years 1714–1718 a series of papers on the summation of the arcs of certain curves, a problem induced by a paper of Johann Bernoulli's of 1698.[1] These papers of Fagnano are reproduced in his *Opere mathematiche* (2 vols.; Albrighi, Segati & Co., Milan, Rome, Naples), II (1911), from which our selection has been translated. In vol. 19 of the *Giornali* Fagnano posed the following problem (*Opere*, II, 271):

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*Problem.* Let a biquadratic primary parabola, which has as its constituent equation $x^4 = y$, and also a portion of it, be given. We ask that another portion of the same curve be assigned such that the difference of the two portions be rectifiable.

---

It had already been recognized by the brothers Bernoulli that what would be called elliptic arcs are not rectifiable, but that sums or differences might be representable by arcs of circles or straight lines. Fagnano gave a solution of his own problem, and generalized it to a number of cases, all involving elliptic integrals. One of his conclusions, sometimes called Fagnano's theorem, dates from 1716 and is found in the paper entitled "Teorema da cui si deduce una nuova misura degli archi elittici, iperbolici, e cicloidali," *Giornali 26* (*Opere*, II, 287–292).

---

*Theorem.* In the two polynomials below, $X$ and $Z$, and in equation (1) the letters $h$, $l$, $f$, $g$ represent arbitrary constant quantities.

I say, in the first place, that if in equation (1) the exponent $s$ expresses the positive unity [$s = +1$], then the integral of the polynomial $X - Z$ is equal to $-hxz/\sqrt{-fl}$.

I say, in the second place, that if in the same equation (1) the exponent $s$ expresses the negative unity [$s = -1$], then the integral of

$$X + Z = \frac{xz\sqrt{-h}}{\sqrt{g}}.$$

Here

$$X = \frac{dx\sqrt{hx^2 + l}}{\sqrt{fx^2 + g}},$$

$$Z = \frac{dz\sqrt{hz^2 + l}}{\sqrt{fz^2 + g}},$$

$$(fhx^2z^2)^s + (flx^2)^s + (flz^2)^s + (gl)^s = 0. \tag{1}$$

---

[1] An account of the contributions of Fagnano to this problem can be found in Cantor, *Geschichte*, III (2nd. ed., 1901), 465–472. Johann Bernoulli's paper, entitled "Theorema universale rectificationi linearum curvarum inserviens" (Universal theorem useful for the rectification of curved lines), appeared in the *Acta Eruditorum* of October 1698 (*Opera omnia*, I, 249–253); in it he asked whether there are curves with arcs that are not rectifiable, but are such that sums or differences of arcs are rectifiable. He claims that the parabola $3a^2y = x^3$ has that property. See Selection V.10, note 4.

The first part of the theorem Fagnano applies to the difference of arcs of an ellipse and of a cycloid, the second part to the sum of arcs of a hyperbola.

Then, in another article, "Metodo per misurare la lemniscata," *Giornali 29* (1718; *Opere*, II, 293–313), he applied his considerations to the lemniscate, a curve discovered by Jakob Bernoulli in 1694.[2] After a reference to the two brothers Bernoulli, Fagnano continues:

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Let the lemniscate be $CQACFC$ [Fig. 1], its semiaxis $CA = a$; then it is known that if we take the origin of the abscissa ($x$) at the center $C$ and call ($y$) the ordinates [*le ordinate*] normal to the axis, then the nature of the lemniscate is

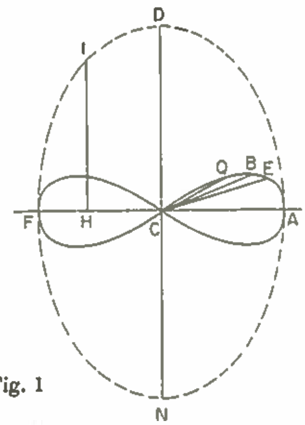


Fig. 1

expressed by this equation: $x^2 + y^2 = a\sqrt{x^2 - y^2}$. It is also known that if we call $z$ the indeterminate chord $CQ = \sqrt{x^2 + y^2}$, then the direct arc

$$CQ = \int \frac{a^2\,dz}{\sqrt{a^4 - z^4}},$$

and the inverse arc

$$QA = \text{arc.}\, CA - \text{arc.}\, CQ = \int \frac{a^2\,dz}{\sqrt{a^4 - z^4}}.$$ [3]

[2] The lemniscate was introduced by Jakob Bernoulli in an article entitled "Constructio curvae accessus et recessus aequabilis" in the *Acta Eruditorum* of September 1694 (*Opera*, II, 608–612) dealing with elastic curves. Here he discusses the curve with equation $xx + yy = a\sqrt{(xx - yy)}$, which curve "of four dimensions" has, as he says, a form "jacentis notae octonari ∞, seu complicatae in nodum fasciae, sive lemnisci, d'un noeud de ruban Gallis" (like a lying eightlike figure, folded in a knot of a bundle, or of a lemniscus, a knot of a French ribbon), *lemniskos* being a knot in the form of an eight. The curve was soon known as a lemniscate.

[3] It was not yet customary to indicate the limits of the integral at the bottom and top of the integral sign, so that the integrals for arcs $CQ$ and $QA$ look alike. Our modern notation $\int_a^b$ is due to J. Fourier; see his *Théorie analytique de la chaleur* (Didot, Paris, 1822), 237–238.

Take the ellipse $ADFNA$, of which the minor semiaxis is $CF = a$, and the major semiaxis $CD = a\sqrt{2}$, and call $z$ the indeterminate abscissa $CH$, with its origin in the center $C$ of the ellipse, and equal to the chord $CQ$ of the lemniscate, and draw the ordinate $HI$ parallel to the major axis. Then it is already known that the direct arc $DI$ of this ellipse has as its expression

$$\int dz \frac{\sqrt{a^2 + z^2}}{\sqrt{a^2 - z^2}},$$

and the inverse arc

$$IF = \text{arc.}\, DF - \text{arc.}\, DI = \int -dz \frac{\sqrt{a^2 + z^2}}{\sqrt{a^2 - z^2}}.$$

Finally, take the equilateral hyperbola $LMP$ with semiaxis $SM = a$ [Fig. 2]. If we call $t$ the indeterminate radius [*applicata*] $SO$, then it is known that if we take the arc $MO$ starting from the center $M$ this arc is expressed as follows:

$$\int \frac{t^2\,dt}{\sqrt{t^4 - a^4}}.$$

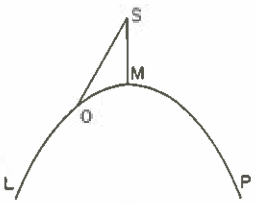

Fig. 2

*Theorem* I. *Let the two equations written below be* (1) *and* (2); *then I say that if we take the first of them, then also the other is valid:*

$$(1) \qquad t = a\frac{\sqrt{a^2 + z^2}}{\sqrt{a^2 - z^2}},$$

$$(2) \qquad \int \frac{a^2\,dz}{\sqrt{a^4 - z^4}} = \int dz \frac{\sqrt{a^2 + zz}}{\sqrt{a^2 - az}} + \int \frac{t^2\,dt}{\sqrt{t^4 - a^4}} - \frac{zt}{a}.$$

The truth of this theorem can be shown by differentiation, and substituting for $t$ and $dt$ their values in terms of $z$ and $dz$ taken from equation (1).

*Corollary*. If in the lemniscate the chord $CQ = z$, and in the ellipse the abscissa $CH$ is also $= z$, and in the equilateral hyperbola $LMP$ the central radius $SO = t$, and if we assign to $t$ its value expressed in equation (1) and substitute in equation (2) the arcs of the curves in terms of their expressions already indicated in the statements above, we obtain

$$\text{arc.}\, CQ = \text{arc.}\, DI + \text{arc.}\, MO = \frac{zt}{a}.$$

Fagnano has in Theorem II another substitution which leads (see Figs. 1 and 2) to

$$\text{arc. } QA = \text{arc. } IF + \text{arc. } ML - \frac{1}{z}\sqrt{a^4 - z^4},$$

and then goes on to

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*Theorem* III. *If we consider equation* (7) *and equation* (8) *below, then I say that, given the first one, the other also is valid:*

$$u = a\frac{\sqrt{a^2 - z^2}}{\sqrt{a^2 + z^2}}, \tag{7}$$

$$\int \frac{a^2\,dz}{\sqrt{a^4 - z^4}} = \int - \frac{a^2\,du}{\sqrt{a^4 - u^4}}. \tag{8}$$

---

From these equations Fagnano again derives some expressions for the arc. Other pairs are (*Opere*, II, 304–309):

$$x = \frac{\sqrt{1 \mp \sqrt{1 - z^4}}}{2}, \qquad \frac{\pm dz}{\sqrt{1 - z^4}} = \frac{dx\sqrt{2}}{\sqrt{1 + x^4}},$$

$$x = \frac{\sqrt{1 \mp z}}{\sqrt{1 \pm z}}, \qquad \frac{\mp dz}{\sqrt{1 - z^4}} = \frac{dx\sqrt{2}}{\sqrt{1 + x^4}},$$

(9) $$\frac{u\sqrt{2}}{\sqrt{1 - u^4}} = \frac{1}{z}\sqrt{1 - \sqrt{1 - z^4}}, \qquad \frac{dz}{\sqrt{1 - z^4}} = \frac{2\,du}{\sqrt{1 - u^4}},$$ (10)

(12) $$\frac{\sqrt{1 - t^4}}{t\sqrt{2}} = \frac{1}{z}\sqrt{1 - \sqrt{1 - z^4}}, \qquad \frac{dz}{\sqrt{1 - z^4}} = \frac{-2\,dt}{\sqrt{1 - t^4}}.$$ (13)

The last equations allow Fagnano to duplicate an arc of the lemniscate, and so to divide the quadrant of the lemniscate into three equal parts; $t = z$ then gives $z = \sqrt[4]{-3 + 2\sqrt{3}}$. He also shows how to divide the quadrant into five equal parts.

Two more sets of equations show how to duplicate an arc of the lemniscate. Fagnano concludes that he can divide the quadrant of the lemniscate therefore into $2 \times 3^m$, $3 \times 2^m$, $5 \times 2^m$ equal parts. "And this is a new and singular property of my curve."

Much later, Fagnano republished his papers in his *Produzioni matematiche* (Pesaro, 1750; reprinted as vol. II of the *Opere matematiche*). When this book reached the Berlin Academy in 1751, Euler, who was asked to express an opinion on it, quickly grasped the importance of Fagnano's transformations for the integration of a number of differential equations of a particular kind, involving radicals. In his "Observationes de comparatione arcuum curvarum ellipticarum," *Novi Commentarii Academiae Scientiarum Petropolitanae 6, 1756–57*

(1761), 58–84 (*Opera omnia*, ser. I, vol. 20, 80–107), he took up, in his own way, Fagnano's investigations on the arcs of the ellipse, the hyperbola, and the lemniscate. In chapter I he sets up formulas on sums and differences of the arcs of the ellipse, in chapter II of the hyperbola, then in chapter III he takes up analogous problems for the case of the lemniscate $(xx + yy)^2 = xx - yy$.

---

*Theorem 4. If, in the lemniscatic curve that we have described here* [Fig. 3] *we draw a chord $CM = z$ and another one besides which is*[4]

$$CN = u = \sqrt{\frac{1 - zz}{1 + zz}},$$

*then the arc $CM$ is equal to the arc $AN$, or also: the arc $CN$ is equal to the arc $AM$.*

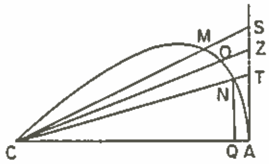


Fig. 3

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The demonstration is like that of Fagnano in a similar case. In Corollary 1 Euler writes $CN = CA\sqrt{\frac{CA^2 - CM^2}{CA^2 + CM^2}}$, in Corollary 2 he changes $u = \sqrt{\frac{1 - zz}{1 + zz}}$ into $z = \sqrt{\frac{1 - uu}{1 + uu}}$ and those expressions into $uuzz + uu + zz = 1$, "hence the points $M$ and $N$ can be interchanged, from which it follows that arc $CM$ = arc $AN$ as well as arc $CN$ = arc $AM$."

Corollary 3 states that, since $CQ$, the abscissa of $N$, is equal to $u\sqrt{\frac{1 + uu}{2}}$ and $QN$, its ordinate, to $u\sqrt{\frac{1 - uu}{2}}$, therefore $CQ = \frac{u}{1 + zz}$, $QN = \frac{uz}{1 + zz}$, and hence $QN/CQ = z$, and $AT = z = CM$ ($AT$ is the tangent at $A$).

Corollary 6 points out that the point $O$, which divides the whole quadrant $CA$ into two equal parts, also divides all arcs $MN$ into two equal parts.

---

*Theorem 5. If in a lemniscatic curve with axis $CA = 1$ we construct* [Fig. 4] *one chord $CM = z$ and another arc besides which is*

$$CM^2 = u = \frac{2z\sqrt{1 - z^4}}{1 + z^4},$$

*then the arc $CM^2$ subtended by this chord $u$ is twice the arc subtended by chord $CM$.*

[4] Compare Fagnano's case (7).

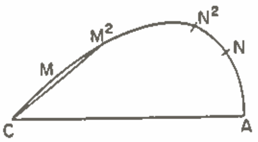


Fig. 4

The demonstration leads, via $uu = \frac{4zz - 4z^6}{1 + 2z^4 + z^8}$, through

$$\sqrt{(1 - uu)} = \frac{1 - 2zz - z^4}{1 + z^4}, \quad \sqrt{(1 + uu)} = \frac{1 + 2zz - z^4}{1 + z^4}, \quad \sqrt{(1 - u^4)} = \frac{1 - 6z^4 + z^8}{(1 + z^4)^2}.$$

and

$$du = \frac{2\,dz(1 - 6z^4 + z^8)}{(1 + z^4)^2\sqrt{(1 - z^4)}}$$

to

$$\frac{du}{\sqrt{1 - u^4}} = \frac{2\,dz}{\sqrt{1 - z^4}},$$ [5]

or, since arc $CM = \int \frac{dz}{\sqrt{1 - z^4}}$, arc $CM^2 = \int \frac{du}{\sqrt{1 - u^4}}$,

$$\text{arc } CM^2 = 2 \text{ arc } CM + \text{const.};$$

but, since $z = 0$ gives $u = 0$, the constant is zero, so that

$$\text{arc } CM^2 = 2 \text{ arc } CM.$$ [6]

In Corollary 1 to Theorem 5 it is pointed out that, if

$$CN = \sqrt{\frac{1 - zz}{1 + zz}},$$

$$CN^2 = \frac{1 - 2zz - z^4}{1 + 2zz - z^4} = \sqrt{\frac{1 - uu}{1 + uu}},$$

then arc $AN$ = arc $CM$, arc $AN^2$ = arc $CM^2$, arc $AN^2 = 2$ arc $AN$.

[5] Compare Fagnano's case (10), interchanging the letters $u$ and $z$.

[6] See C. L. Siegel, "Zur Vorgeschichte des Eulerschen Additionstheorems," *Sammelband zu Ehren des 250. Geburtstages Leonhard Eulers*, ed. K. Schröder (Akademie Verlag, Berlin, 1959), 315–317.

In Corollary 4 it is pointed out that when $M$ and $N^2$ coincide the arc $CMNA$ is divided into three equal parts. This leads to a fifth-degree equation,

$$(1 + z)(1 - \mu z + zz)(1 + \mu z + zz) = 0,$$

with $\mu = 1 + \sqrt{3}$, hence $CM = \dfrac{1 + \sqrt{3} - \sqrt{2\sqrt{3}}}{2}$, $CN = \sqrt{\dfrac{2\sqrt{3}}{1 + \sqrt{3}}}$.

Other corollaries give formulas for half a given arc and the fifth part of a quadrant; the number of equal parts that can be computed is $2^m(1 + 2^n)$.

---

*Theorem 6. If the chord of a simple arc $CM$ is $z$ and the chord of the n-fold arc $CM^2 = u$, then the chord of the $(n + 1)$-fold arc is*

$$CM^{n+1} = \frac{z\sqrt{\dfrac{1 - uu}{1 + uu}} + u\sqrt{\dfrac{1 - zz}{1 + zz}}}{1 - uz\sqrt{\dfrac{(1 - uu)(1 - zz)}{(1 + uu)(1 + zz)}}}.$$

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In the paper "De integratione aequationis differentialis," *Novi Commentarii Academiae Scientiarum Petropolitanae 6, 1756–57* (1761), 37–57 (*Opera omnia*, ser. I, 20, 58–79), printed in front of the previous paper but written somewhat later, Euler returned, in his own way, to the principle expressed in "Fagnano's theorem," and thereby clarified its character. The full title of the paper reads in translation:

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On the integration of the differential equation

$$\frac{m\,dx}{\sqrt{1 - x^4}} = \frac{n\,dy}{\sqrt{1 - y^4}},$$

---

comparing the case first with that of $\dfrac{m\,dx}{\sqrt{1 - x^2}} = \dfrac{n\,dy}{\sqrt{1 - y^2}}$, which leads to $m \sin^{-1} x = n \sin^{-1} y + C$ (Euler writes $A$ sin for $\sin^{-1}$).

---

*Theorem. I therefore say that of the differential equation*

$$\frac{dx}{\sqrt{1 - x^4}} = \frac{dy}{\sqrt{1 - y^4}}$$

*the complete integral equation is*

$$xx + yy + ccxxyy = cc + 2xy\sqrt{1 - c^4}.$$

*Demonstration.* When we take this equation its differential will be

$$x\,dx + y\,dy + ccxy(x\,dy + y\,dx) = (x\,dy + y\,dx)\sqrt{1 - c^4},$$

from which we obtain

$$dx[x + ccxyy - y\sqrt{(1 - c^4)}] + dy[y + ccxxy - x\sqrt{(1 - c^4)}] = 0.$$

Solving the same equation we obtain

$$y = \frac{x\sqrt{(1 - c^4)} + c\sqrt{(1 - x^4)}}{1 + ccxx} \quad \text{and} \quad x = \frac{y\sqrt{(1 - c^4)} - c\sqrt{(1 - y^4)}}{1 + ccyy}.$$

If we now assign to the radical $\sqrt{(1 - x^4)}$ the sign $+$, we must assign to the radical $\sqrt{(1 - y^4)}$ the sign $-$, so that the value $x = 0$ gives in both cases the value $y - c$. Therefore we have

$$x + ccxyy - y\sqrt{(1 - c^4)} = -c\sqrt{(1 - y^4)},$$
$$y + ccxxy - x\sqrt{(1 - c^4)} = c\sqrt{(1 - x^4)}.$$

When we substitute these values in the differential equation, we obtain

$$-c\,dx\sqrt{(1 - y^4)} + c\,dy\sqrt{(1 - x^4)} = 0,$$

or

$$\frac{dx}{\sqrt{(1 - x^4)}} = \frac{dy}{\sqrt{(1 - y^4)}}.$$

The integral of this differential equation is therefore

$$xx + yy + ccxxyy = cc + 2xy\sqrt{(1 - c^4)}.$$

and, since it contains the arbitrary constant $c$, it is the complete integral. Q.E.D.

10. If, therefore, we have the equation

$$\frac{dx}{\sqrt{(1 - x^4)}} = \frac{dy}{\sqrt{(1 - y^4)}},$$

then the complete value of the integral in $x$ is

$$x = \frac{y\sqrt{(1-c^4)} \pm c\sqrt{(1-y^4)}}{1+ccyy},$$

which passes into $x = y$ if the constant $c$ vanishes, and if we place $c = 1$ we obtain

$$x = \pm\frac{\sqrt{(1-y^4)}}{1+yy} = \sqrt{\frac{1-yy}{1+yy}},$$

which are both particular values already found above [in §9]. From here we obtain other particular values, but which lead to imaginaries. Thus if we take $c = 0$ we obtain

$$x = \frac{\sqrt{-1}}{y},$$

and if we take $cc = -1$ we obtain

$$x = \sqrt{\frac{yy+1}{yy-1}},$$

which also satisfy the equation in question.

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## 19 EULER, LANDEN, LAGRANGE. THE METAPHYSICS OF THE CALCULUS

Many eighteenth-century mathematicians tried to give a solid foundation to the calculus. We present here three of these attempts. Euler, in his *Institutiones calculi differentialis* (Saint Petersburg, 1755; *Opera omnia*, ser. I, vol. 10), gave his theory of the zeros of different orders, $dx$ being, he said, equal to 0. John Landen (1719–1790), an English surveyor and land agent, best remembered because of his contributions to the theory of elliptic integrals, defined his derivative by the "residue" $\left[\frac{f(x_1)-f(x_0)}{x_1-x_0}\right]_{x_0=x_1}$, expanding $f(x)$ in a power series in $x$ (concentrating on the binomial theorem). We find this in the *Discourse concerning the residual analysis* (London, 1758). A few years later, Lagrange, in his "Note sur la métaphysique du calcul infinitésimal," *Miscellanea Taurinensia 2* (1760–61), reprinted in *Oeuvres*, V (1877), 597–599, gave what he thought to be an improvement on Landen's "algebraic" method, basing his whole comprehensive reevaluation of the principles of the calculus on the Taylor expansion. Lagrange later gave a full exposition in his *Théorie des fonctions analytiques* (Paris, 1797), of which the second edition, revised (1813), is reprinted in *Oeuvres*, IX (1881).

Euler's method has long been rejected, often with a kind of shoulder shrugging indicating that even the great Euler sometimes slept. A more appreciative note has recently been struck by A. P. Juschkewitch, "Euler und Lagrange über die Grundlagen der Analysis,"