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FERMAT

On Analytic Geometry

(Translated from the French by Professor Joseph Seidlin, Alfred College, Alfred, N. Y.)

The following extract is from Fermat's *Introduction aux Lieux Plans et Solides*. It appears in the the *Varia Opera Mathematica* of Fermat in 1679, and in the *Œuvres de Fermat*, ed. Tannery and Henry, Paris, 1896. It shows how clearly Fermat understood the connection between algebra and geometry. It will be observed that Fermat uses the terms "plane and solid loci" in an older sense, somewhat different from the one now recognized.

The French text will be found in the *Œuvres*, vol. III, pp. 85–96.

Introduction to Plane and Solid Loci

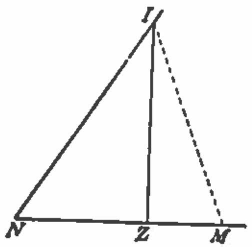
None can doubt that the ancients wrote on loci. We know this from Pappus, who, at the beginning of Book VII, affirms that Apollonius had written on plane loci and Aristæus on solid loci. But, if we do not deceive ourselves, the treatment of loci was not an easy matter for them. We can conclude this from the fact that, despite the great number of loci, they hardly formulated a single generalization, as will be seen later on. We therefore submit this theory to an apt and particular analysis which opens the general field for the study of loci.

Whenever two unknown magnitudes appear in a final equation, we have a locus, the extremity of one of the unknown magnitudes describing a straight line or a curve. The straight line is simple and unique; the classes of curves are indefinitely many,—circle, parabola, hyperbola, ellipse, etc.

When the extremity of the unknown magnitude which traces the locus, follows a straight line or a circle, the locus is said to be plane; when the extremity describes a parabola, a hyperbola, or an ellipse, the locus is said to be solid. . . .

It is desirable, in order to aid the concept of equation, to let the two unknown magnitudes form an angle, which usually we would suppose to be a right angle, with the position and the extreme point of one of the unknown magnitudes established. If neither of the two unknowns is greater than a quadratic, the locus will be plane or solid, as can be clearly seen from the following:

Let NZM be a straight line of given position with point N fixed. Let NZ be the unknown quantity a and ZI (the line drawn to form the angle NZI) the other unknown quantity e.



If $da = be$, the point I will describe a line of fixed position. Indeed, we would have $\frac{b}{d} = \frac{a}{e}$. Consequently the ratio $a:e$ is given, as is also the angle at Z. Therefore both the triangle NIZ and the angle INZ are determined. But the point N and the position of the line NZ are given, and so the position of NI is determined. The synthesis is easy.

To this equation we can reduce all those whose terms are either known or combined with the unknowns a and e, which may enter simply or may be multiplied by given magnitudes.

$$z^{II} - da = be.$$

Suppose that $z^{II} = dr$. We then have

$$\frac{b}{d} = \frac{r - a}{e}.$$

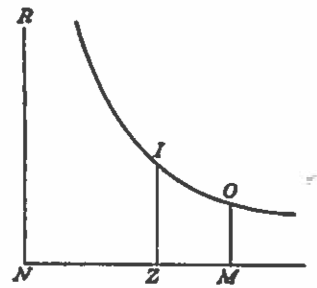
If we let $MN = r$, point M will be fixed and we shall have $MZ = r - a$.

The ratio $\frac{MZ}{ZI}$ therefore becomes fixed. With the angle at Z given, the triangle IZM will be determined, and in drawing MI it follows that this line is fixed. Thus point I will be on a line of determined position. A ike conclusion can be reached without difficulty for any equation containing the terms a or e.

Here is the first and simplest equation of a locus, from which all the loci of a straight line may be found; for example, the proposition 7 of Book I of Apollonius "On Plane Loci," which has since, however, found a more general expression and mode of construction. This equation yields the following interesting proposition: "Assume any number of lines of given position. From a given point draw lines forming given angles. If the sum of the products of the lines thus drawn by the given lines equals a given area, then the given point will trace a line of determined position."

We omit a great number of other propositions, which could be considered as corollaries to those of Apollonius.

The second species of equations of this kind are of the form $ae = z^{II}$, in which case point I traces a hyperbola. Draw NR parallel to ZI; through any point, such as M, on the line NZ, draw MO parallel to ZI. Construct the rectangle NMO equal in area to z^{II}. Through the point O, between the asymptotes NR, NM, describe a hyperbola; its position is determined and it will pass through point I, having assumed, as it were, ae,—that is to say the rectangle NZI,—equivalent to the rectangle NMO. To this equation we may reduce all those whose terms are in part constant, or in part contain a or e or ae.



If we let

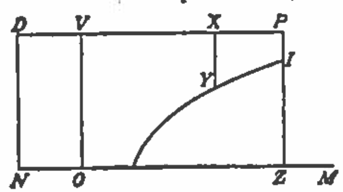
$$d^{II} + ae = ra + se$$

we obtain by fundamental principles $ra + se - ae = d^{II}$. Construct a rectangle of such dimensions as shall contain the terms $ra + se - ae$. The two sides will be $a - s$ and $r - e$, and their rectangle, $ra + se - ae - rs$.

If from d^{II} we subtract rs, the rectangle

$$(a - s)(r - e) = d^{II} - rs.$$

Take NO equal to s, and ND, parallel to ZI, equal to r. Through point D, draw DP parallel to NM; through point O, OV parallel to ND; prolong ZI to P.

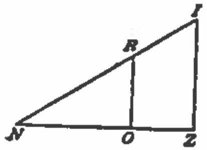


Since $NO = s$ and $NZ = a$, we have $a - s = OZ = VP$. Similarly, since $ND = ZP = r$ and $ZI = e$, we have $r - e = PI$. The rectangle $PV \times PI$ is therefore equal to the given area $d^{II} - rs$; the point I is therefore on a hyperbola having PV, VO as asymptotes.

If we take any point X, the parallel XY, and construct the rectangle $VXY = d^{II} - rs$, and through point Y we describe a hyperbola between the asymptotes PV, VO, it will pass through point I. The analysis and construction are easy in every case.

The following species of loci equations arises if we have $a^2 = e^2$ or if a^2 is in a given relation to e^2, or, again, if $a^2 + ae$ is in a given relation to e^2. Finally this type includes all the equations whose terms are of the second degree containing a^2, e^2, or ae. In all these cases point I traces a straight line, which is easily demonstrated.

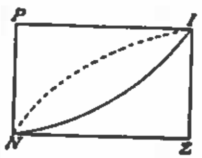
If the ratio $\frac{NZ^2 + NZ.ZI}{ZI^2}$ is given, and any parallel OR is drawn, then it is easy to show that $\frac{NO^2 + NO.OR}{OR^2}$ has the value of the given ratio. The point I will therefore be on a line of determined position. The same will be true of all equations whose terms are either the squares of the unknowns or their product. It is needless to enumerate additional specific instances.



If to the squares of the unknowns, with or without their product, are added absolute terms or terms which are the products of one of the unknowns by a given magnitude, the construction is more difficult. We shall indicate the construction and give the proof for several cases.

If $a^2 = de$, point I is on a parabola.

Let NP be parallel to ZI; with NP as diameter, construct the parabola whose parameter is the given line d and whose ordinates are parallel to NZ. The point I will be on the parabola whose position is defined. In fact, it follows from the construction that the rectangle $d \times NP = PI^2$, that is, $d \times IZ = NZ^2$ and, consequent y, $de = a^2$.



To this equation we can easily reduce all those in which, with a^2, appear the products of the given magnitudes and e, or with e^2 appear the products of the given magnitudes with a. The same would hold true were the equation to contain absolute terms.

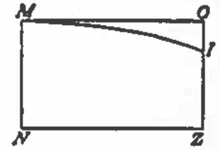
If, however, $e^2 = da$, then, in the preceding figure, with N as vertex and with NZ as diameter, construct the parabola whose parameter is d and whose ordinates are parallel to the line NP. It is plain that the imposed condition is satisfied.

If we let $b^2 - a^2 = de$, we have $b^2 - de = a^2$. Divide b^2 by d; let $b^2 = dr$, and we have $dr - de = a^2$ or $d(r - e) = a^2$.

We shall have reduced this equation to the former [—that is, $a^2 = de$,—] by replacing $r - e$ by e.

Let us assume MN (p. 393) parallel to ZI and equal to r; through the point M draw MO parallel to NZ. Point M and the position of the line MO are now given. It follows from the construction that $OI = r - e$. Therefore $d \times OI = NZ^2 = MO^2$.

The parabola drawn with M as vertex, diameter MN, d as parameter, and the ordinates parallel to NZ, satisfies the condition as is clearly shown by the construction.

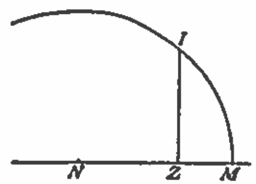


If $b^2 + a^2 = de$, we have $de - b^2 = a^2$, etc., as above. Similarly then we can construct all the equations containing a^2 and e.

But a^2 is often found with e^2 and with absolute terms. Let $b^2 - a^2 = e^2$.

The point I will be on a circle of determined position if the angle NZI is a right angle.

Assume MN equal to b. The circle described with N as center and with NM as radius will satisfy the condition. That is to say, that no matter which point I is taken, anywhere on the circumference, it is clear that ZI^2 (or e^2) will equal NM^2 (or b^2) $- NZ^2$ (or a^2).



To this equation may be reduced all those containing terms in a^2, e^2, and in a or e multiplied by given magnitudes, provided angle NZI be a right angle, and, moreover, that the coefficient of a^2 be equal to that of e^2.

Let

$$b^2 - 2da - a^2 = e^2 + 2re.$$

Adding r^2 to both sides and, thus replacing e by $e + r$, we have

$$r^2 + b^2 - 2da - a^2 = e^2 + r^2 + 2re.$$

Adding d^2 to $r^2 + b^2$, thus replacing a by $d + a$, and denoting the sum of the squares $r^2 + b^2 + d^2$ by p^2, we get

$$p^2 - d^2 - 2da - a^2 = r^2 + b^2 - 2da - a^2,$$

which leads to

$$p^2 - d^2 = r^2 + b^2.$$

If now we replace $a + d$ by a and $e + r$ by e, we shall have

$$p^2 - a^2 = e^2,$$

which equation is reduced to the preceding.

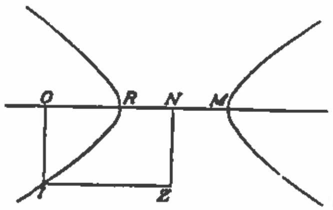
By like reasoning we are able to reduce all similar equations. Based on this method we have built up all of the propositions of the Second Book of Apollonius "On Plane Loci" and we have proved that the six first cases have loci for any points whatever, which is quite remarkable and which was probably unknown to Apollonius.

When $\frac{b^2 - a^2}{e^2}$ is a given ratio, the point I will be on an ellipse. Let MN equal b. With M as vertex, NM as diameter, and N as center describe an ellipse whose ordinates are parallel to ZI, so that the squares of the ordinates shall be in a given ratio to the product of the segment of the diameter. The point I will be on that ellipse. That is, $NM^2 - NZ^2$ is equal to the product of the segments of the diameter.

To this equation can be reduced all those in which a^2 is on one side of the equation and e^2 with an opposite sign and a different coefficient on the other side. If the coefficients are the same and the angle a right angle, the locus will be a circle, as we have said. If the coefficients are the same but the angle is not a right angle, the locus will be an ellipse.

Moreover, though the equations include terms which are products of a or e by given magnitudes, the reduction may nevertheless be made by the method which we have already employed.

If $(a^2 + b^2):e^2$ is a given ratio, the point I will be on a hyperbola. Draw NO parallel to ZI; let the given ratio be equal to $b^2:NR^2$. Point R will then be fixed. With R as vertex, RO as diameter, and N as center, construct an hyperbola whose ordinates are parallel to NZ, such that the product of the whole diameter (MR) by RO together with RO^2 shall be to OI^2 as $NR^2:b^2$. It follows, letting $MN = NR$, that $(MO \times OR + NR^2):(OI^2 + b^2)$ is equal to $NR^2:b^2$, the given ratio.



But

$$MO \times OR + NR^2 = NO^2 = ZI^2 = e^2$$

and

$$OI^2 + b^2 = NZ^2 \text{ (or } a^2) + b^2.$$

Therefore $e^2:(b^2 + a^2) = NR^2:b^2$ and, inverting, $(b^2 + a^2):e^2$ is the given ratio. Therefore point I is on an hyperbola of determined position.

By the scheme we have already employed we may reduce to this equation all those in which a^2 and e^2 are contained with given terms (separately) or with expressions involving the products of a or e by the given terms, and in which a^2 and e^2 have the same sign and appear on the opposite sides of the equation. If the signs were different the locus would be a circle or an ellipse.

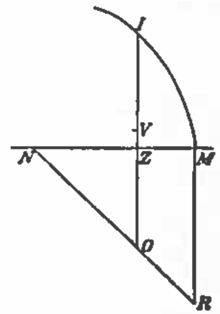
The most difficult type of equation is that containing, along with a^2 and e^2, terms involving ae, other given magnitudes, etc. Let

$$b^2 - 2a^2 = 2ae + e^2.$$

Add a^2 to both sides so as to have $a + e$ as a factor of one of the members. Then

$$b^2 - a^2 = a^2 + 2ae + e^2.$$

Replace $a + e$ by, say, e; then, according to the preceding development, the circle MI will satisfy the equation; that is to say, MN^2 $(=b^2) - NZ^2$ $(=a^2) = ZI^2(=[a + e]^2)$. Letting $VI = NZ$ $= a$, we have $ZV = e$.



In this problem, however, we are looking for the point V or the extremity of the line e. It is therefore necessary to find, and to indicate, the line upon which the point V is located.

Let MR be parallel to ZI and equal to MN. Draw NR which meets IZ, prolonged, at O. Since $MN = MR$, $NZ = ZO$. But $NZ = VI$; therefore, by addition, $VO = ZI$. Therefore $MN^2 - NZ^2 = VO^2$. But triangle NMR is known; therefore the ratio $NM^2:NR^2$ is given as are also the ratios $NZ^2:NO^2$ and $(MN^2 - NZ^2):(NR^2 - NO^2)$. But we have proved that $OV^2 = MN^2 - NZ^2$. Therefore the ratio $(NR^2 - NO^2):OV^2$ is known. But the points N and R are given, as well as the angle NOZ. Therefore, as we have just shown, point V is on an ellipse.

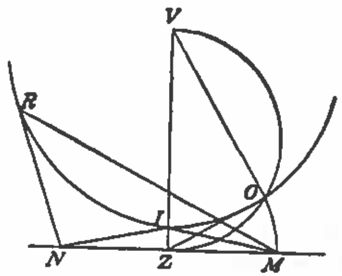
By analogous procedure we reduce to the preceding cases all the others in which along with the terms containing ae and a^2 or e^2 are also terms consisting of products of a and e by given magnitudes. The discussion of these different cases is very easy. The problem may always be solved by means of a triangle of known configuration.

We have therefore included in a brief and clear exposition all that the ancients have left unexplained concerning plane and solid loci. Consequently one can recognize at once which loci apply to all cases of the final proposition of Book I of Apollonius "On Plane Loci," and one can generally discover without great difficulty all which pertains to that matter.

As a culminating point to this treatise, we shall add a very interesting proposition of almost obvious simplicity:

"Given the position of any number of lines; if from some definite point lines be drawn forming given angles with the given lines, and the sum of the squares of all the segments is equal to a given area, the point will describe a solid locus of determined position."

A single example will suffice to indicate the general method of construction. Given two points N and M, required the locus of the points such that the sum of the squares of IN, IM, shall be in a given ratio to the triangle INM.



Let $NM = b$. Let e be the line ZI drawn at right angles to NM, and let a be the distance NZ. In accordance with fundamental principles, $(2a^2 + b^2 - 2ba + 2e^2) : be$ is a given ratio. Following in treatment the procedures previously explained we have the suggested construction.

Bisect NM at Z; erect at Z the perpendicular ZV; make the ratio $4ZV:NM$ equal to the given ratio. On VZ draw the semicircle VOZ, inscribe $ZO = ZM$, and draw VO. With V as center and VO as radius draw the circle OIR. If from any point R on this circle, we draw RN, RM, I say that $RN^2 + RM^2$ is in the given ratio to the triangle RNM.

The constructions of the theorems on loci could have been much more elegantly presented if this discovery had preceded our already old revision of the two books on plane loci. Yet, we do not regret this work, however precocious or insufficiently ripe it may be. In fact, there is for science a certain fascination in not exposing to posterity works which are as yet spiritually incomplete; the labor of the work at first simple and clumsy gains strength as well as stature through new inventions. It is quite important that the student should be able to discern clearly the progress which appears veiled as well as the spontaneous development of the science.