Astronomical calculations workshop[1]

Introduction The increase of observational accuracy in 16th century astronomy prompted computations with more significant digits, which took more time to perform. Computational aids were developed to speed up the work. In this workshop you will experience this development yourself. You will get new views on computation without calculators, and understand the role of astronomy in the development of computational techniques, and maybe also appreciate the work of "computers" when that word still meant "people of flesh and blood doing computations".

As an example we will take the computation of the position of the sun in the ecliptic[2] from the observed sun's altitude at noon (when it is in the meridian of the place).

Data We are in Utrecht at latitude $\phi = 52°$ and assume that at noon, when the sun is due south and at its highest position, we observe the altitude h of the centre of the sun above the horizon. We calculate the sun's declination δ from the formula $\delta = h - (90° - \phi)$. Declination is for the sun what latitude is for places on earth: it is the angular distance to the equator. Declination is North (or positive) in spring and summer, South (negative) in fall and winter. In the equinoxes, when the sun on the ecliptic passes the equator, it is zero. This happens approx. at 21 march and 21 sept.

We also need the angle between the planes of the ecliptic and the equator ε. This ε is also the angle between the earth's axis and a perpendicular to the ecliptic plane. Its value is $\varepsilon \approx 23°30'$ as can be observed by noting the maximum (or minimum) value that the declination reaches annually, i.e., at the summer (or winter) solstice.

Place in the ecliptic With the help of trigonometric tables to look up sines, we can compute

$$\sin \lambda^\star = \frac{\sin |\delta|}{\sin \epsilon}$$

where $\lambda^\star$ denotes an angle between 0 and 90°. The position of the sun in the ecliptic λ is then found by:

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[1] Adapted from an earlier version of Jan P. Hogendijk.

[2] The ecliptic is the apparent path of the sun against the background of the stars. In the heliocentric world view it is the plane of the earth around the sun. Mesopotamian astronomers have divided the ecliptic in 12 signs of 30° each. We still name these signs by the (Greek) names of the constellations Aries, Taurus, etc., where 0° Aries corrsponds with the vernal equinox (= intersection point of the ecliptic and the celestial equator). However, this intersection point has drifted backwards almost a full sign due to precession of the earth's axis. Therefore the sun is actually in the constellation of Pisces when we say that it is in the sign of Aries.

$\lambda = \lambda^\star$ between 21/3 and 21/6 (i.e., spring, between vernal equinox and summer solstice, or δ increasing between 0 and ε);

$\lambda = 180° - \lambda^\star$ between 21/6 and 21/9 (summer, δ decreasing from ε to 0);

$\lambda = 180° + \lambda^\star$ between 21/9 and 21/12 (autumn, δ from 0 to $-\varepsilon$);

$\lambda = 360° - \lambda^\star$ between 21/12 and 21/3 (winter, δ from $-\varepsilon$ to 0).

The answer can also be given modulo 30 and with the correct sign (each comprising 30°) Aries ♈, Taurus ♉, Gemini ♊, Cancer ♋, Leo ♌, Virgo ♍, Libra ♎, Scorpio ♏, Sagittarius ♐, Capricorn ♑, Aquarius ♒, Pisces ♓. For example, 55° equals 25° ♉.

To do You will receive a value of $|\delta|$ and a season, and perform the λ-calculation in three different ways **keeping track of the time required for each method**. Also investigate the accuracy of the answer (although note that the tables that we use are more primitive than the contemporary ones).

First method You have received a sine table showing sines for a circle of radius 10000 (or, showing only the first 4 digits of the fractional part of the numbers). Find $\sin|\delta|$ and $\sin\epsilon$ and compute $\frac{\sin|\delta|}{\sin\epsilon}$ by long division in 4 decimal places. This is the sine of $\lambda^\star$. Find $\lambda^\star$ with the sine table and then convert to λ.

How long dit it take? Assuming δ is accurate to $3'$, what is the expected accuracy of your $\lambda^\star$?

Second method Most of the time is spent on the division, so it is worthwhile to work around it. This is all the more true if you would have to do the computation in 6 or 8 decimal places and with more complicated formulas, as was often the case in daily astronomical work.

One workaround is called *prostaphaeresis*,[3] a technique based on the addition/subtraction formulae in trigonometry:

$$\sin p \sin q = \frac{1}{2}\Big(\cos(p-q) - \cos(p+q)\Big) = \frac{1}{2}\Big(\sin(90^o - p + q) - \sin(90^o - p - q)\Big)$$

We can apply the technique only after some preparation. First, note that in our formula for $\lambda^\star$ we divide by $\sin\epsilon \approx 0.4$, or equivalently, we multiply by $1/\sin\epsilon \approx 2.5$. Still better, considering that $\frac{1}{10\sin\epsilon} \approx 0.25$ is a number between 0 and 1, we know that there exists an angle $0 < p < 90°$ such that $\sin p = \frac{1}{10\sin\epsilon}$. With this p and the prostaphaeresis formula we get

$$\sin\lambda^\star = 10 \sin p \sin\delta = \frac{10}{2}\Big(\sin(90° - p + \delta) - \sin(90° - p - \delta)\Big).$$

This has only additions, subtractions and table-lookups, except one trivial multiplication/division. Note that $90° - p$ is in fact a constant.

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[3] A contraction of the Greek words for addition and subtraction.

Now let's do the workshopwork. First, find $90° - p$; this can be done collectively since it is the same constant for everyone. Second, compute λ for your own δ and season. Time the second phase and make a judgement of the accuracy as in the first method.

Third method   Napier published his logarithms in 1614, exactly 400 years ago. In fact, the first logarithmic tables tabulated log sines because astronomers were the prime consumers of this innovative new technique!

Sines are smaller than 1 and therefore log sines are negative. Negative numbers were considered as something to avoid (and indeed they potentially mess up any practical computation, as your own pupils will doubtlessly acknowledge!). Therefore 10 was added to the table values, so actually the function $f(x) = 10 + \log \sin x$ was tabulated.

You have received such a table in 4 decimal places. The integer part (always 7, 8, or 9) of the values is stated only once per table row to save space. A bar above a number signals that the integer part has increased by 1. The base of the logarithm is immaterial as long as you convert to and from logs using tables of the same base.

We make use of this table in the following way. Since

$$\log \lambda^\star = \log \delta - \log \varepsilon$$

we have

$$f(\lambda^\star) = f(\delta) - (f(\varepsilon) - 10) = f(\delta) + (10 - f(\varepsilon)).$$

Note that $10 - f(\varepsilon)$ is again a constant.

Find your λ, time your work, and consider the accuracy of the result.

Finally   When all the computations have been done we will discuss their pros and cons, considering:

- How long does the computation take?

- How accurate is the result, based on 4-digit tables? What if we take tables with more digits?

- Does the method work equally well for all values of δ?