Trigonometry

1. China is a big country, and it has bamboos in it. This is reflected in their methods for measuring the heavens. As Chen Zi says, "16,000 li to the south at the summer solstice, and 135,000 li to the south at the winter solstice, if one sets up a post at noon it casts no shadow. This single [fact is the basis of] the numbers of the Way of Heaven." (From *The book of Chen Zi*, in the *Mathematical Classics of the Zhou Gnomon*, compiled around the first century BCE.)

   Chen Zi is referring to the point S' on the earth's surface perpendicularly beneath the sun S. Now, standing somewhere else, we erect a bamboo tube BB' so that its shadow falls at our feet O.

   (a) Draw a picture of this and use it to find a formula for the distance to the sun from the earth, SS', in terms of measurable quantities. (Assume that the earth is flat.)

   Now, Chen Zi continues, pick up the bamboo tube you used as the post and point it towards the sun. Suppose its diameter is just big enough for you to see the whole sun through the tube (otherwise go get a longer or shorter tube).

   (b) Draw a picture of this and use it to find a formula for the diameter of the sun in terms of measurable and known quantities.

   (c) The value for SS' reported by Chen Zi is 80,000 li. Does this seem accurate? (Rather than trying to look up how long a li is you can answer this on the basis of the information in the quotation above.)

2. While the Chinese thus utilised their benefit of having vast land and bamboo sticks at their disposal, the Muslims faced other circumstances in response to which they devised other methods. Al-Biruni (*Book of the Determination of the Coordinates of Localities*, c. 1025, chapter 5) discusses a method for measuring the circumference of the earth akin to that of Eratostenes, but does not find it feasible:

   "Who is prepared to help me in this [project]? It requires strong command over a vast tract of land and extreme caution is needed from the dangerous treacheries of those spread over it. I once chose for this project the localities between Dahistan, in the vicinity of Jurjan and the land of the Chuzz (Turks), but the findings were not encouraging, and then the patrons who financed the project lost interest in it."

   Instead: "Here is another method for the determination of the circumference of the earth. It does not require walking in deserts."

   The method is this. Climb a mountain. Let M be the mountain top, E its base, and C the centre of the earth. Now look towards the horizon and let H be the point furthest away from you that you can see.

   (a) Draw a picture of this and use it to find a formula for the radius of the earth in terms of measurable quantities.

   Al-Biruni did indeed carry this out:

   "When I happened to be living in the fort of Nandana in the land of India, I observed from an adjacent high mountain standing west from the fort, a large plain lying south of the mountain. It occurred to me that I should examine this method there. So, from the top of the mountain, I made an empirical measurement of the contact between the earth and the blue sky. I found that the line of sight had dropped below the reference line by the amount 0;34°. Then I measured the perpendicular of the mountain and found it to be 652;3,18 cubits, where the cubit is a standard of length used in that region for measuring cloth." Al-Biruni goes on to calculate the radius of the earth from this data, which comes out as 12,803,337;2,9 cubits.

These numbers are given in a mixed notation. The integer part (before the semicolon) is given in ordinary decimal notation, while the fractional part (after the semicolon) is given in sexagesimal (base 60) form. Thus, for example, $12,803,337;2,9 = 12803337;2,9 = 12803337 + \frac{2}{60} + \frac{9}{60^2}$.

(b) Check al-Biruni's calculation. Note that some discrepancy results from imperfections in the trigonometric tables available to him.

"Cubit" means "forearm," which makes sense as a unit for "measuring cloth." Although the exact value of a cubit intended by an author is often unclear, one may generally assume it to be about 44 cm.

(c) In terms of metric units, with what accuracy does al-Biruni give the height of the mountain?

The downside of this method is of course that it requires "a high mountain close to the seashore, or close to a large level desert." Thus, coming across such a mountain is an opportunity too good to pass up even if you are in the middle of a war:

"Abu al-Tayyib Sanad bin 'Ali has narrated that he was in the company of al-Ma'mun when he made his campaign against the Byzantines, and that on his way he passed by a high mountain close to the sea. Then al-Ma'mun summoned him to his presence and ordered him to climb that mountain, and to measure at its summit the dip of the sun."

Why did the Chinese measure the heavens and the Muslims the earth? Shang Gao expresses the Chinese attitude: "he who knows Earth is wise, but one who knows Heaves is a sage." But in the Muslim world, different cultural circumstances conferred a higher status on earth-measurements:

"If the investigation of distances between towns, and the mapping of the habitable world, ... serve none of our needs except the need for correcting the direction of the qibla we should find it our duty to pay all our attention and energy for that investigation. The faith of Islam has spread over most parts of the earth, and its kingdom has extended to the farthest west; and every Muslim has to perform his prayers and to propagate the call of Islam for prayer in the direction of the qibla." (al-Biruni, *ibid.*)

Greek trigonometry also reflects its cultural context. In the age of Socrates, Plato, and Aristotle, Athens was the cultural center. A new era of Hellenistic culture, however, was initiated by the conquests Alexander the Great. His wars spread Greek culture to Egypt among other places, where one of the cities named after him, Alexandria, was to become the new intellectual capital. Aristotle went to Macedonia to teach the young Alexander in year -343, thus marking the boundary of the two eras. Euclid wrote his *Elements* in Alexandria around year -300, synthesising great amounts of "pure" mathematics in the classical Athenian style. Later Hellenistic mathematics tends to be more "applied," broadly speaking, perhaps in part triggered by the logistic requirements of a rapidly expanding empire. The fact that Eratostenes' measurement of the size of the earth involves a well can thus be seen as a contextual indicator analogous to Chinese and Islamic cases above.

3. Another example of logistically motivated trigonometry is Heron's method for digging a tunnel through a mountain from both ends simultaneously. We wish to dig a straight tunnel from a point N to the north of a mountain to a point S to the south of a mountain. How can we make sure that the diggers starting at either end meet in the middle instead of digging past each other and making two tunnels? Of course the diggers cannot see their goal point. Solve this problem by introducing a point E to the east of the mountain that is visible from both N and S, and then use trigonometry to arrive at a practical way of determining the directions of digging.